REACTION DIFFUSION EQUATIONS WITH SUPER-LINEAR ABSORPTION: UNIVERSAL BOUNDS, UNIQUENESS FOR THE CAUCHY PROBLEM, BOUNDEDNESS OF STATIONARY SOLUTIONS

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Abstract.

Consider classical solutions $u\in C^2(R^n\times(0,\infty))\cap C(R^n\times[0,\infty))$ to the parabolic reaction diffusion equation

$$\begin{aligned}&u_t=Lu+f(x,u),\ (x,t)\in R^n\times(0,\infty);\\&u(x,0)=g(x)\ge 0,\ x\in R^n;\\&u\ge 0,\end{aligned}$$

where

$$L=\sum_{i,j=1}^n a_{i,j}(x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^n b_i(x)\frac{\partial}{\partial x_i}$$

is a non-degenerate elliptic operator, $g\in C(R^n)$ and the reaction term f converges to $-\infty$ at a super-linear rate as $u\to\infty$. We give a sharp minimal growth condition on f, independent of L, in order that there exist a universal, a priori upper bound for all solutions to the above Cauchy problem—that is, in order that there exist a finite function $M(x,t)$ on $R^n\times(0,\infty)$ such that $u(x,t)\le M(x,t)$, for all solutions to the Cauchy problem. Assuming now in addition that $f(x,0)=0$, so that $u\equiv 0$ is a solution to the Cauchy problem, we show that under a similar growth condition, an intimate relationship exists between two seemingly disparate phenomena—namely, uniqueness for the Cauchy problem with initial data $g=0$ and the nonexistence of unbounded, stationary solutions to the corresponding elliptic problem. We also give a generic condition for nonexistence of nontrivial stationary solutions.

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1. Introduction and statement of results. Consider classical solutions $u \in C^2(R^n \times (0,\infty)) \cap C(R^n \times [0,\infty))$ to the parabolic reaction diffusion equation

$$u_t = Lu + f(x,u),\ (x,t) \in R^n \times (0,\infty);$$

$$u(x,0) = g(x) \geq 0,\ x \in R^n; \tag{1.1}$$

$$u \geq 0,$$

where

$$L = \sum_{i,j=1}^n a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial}{\partial x_i},$$

with $a_{i,j}, b_i \in C^\alpha(R^n)$ and $\{a_{i,j}\}$ strictly elliptic; that is, $\sum_{i,j=1}^n a_{i,j}(x)\nu_i\nu_j > 0$, for all $x \in R^n$ and $\nu \in R^n - \{0\}$. We assume that $g \in C(R^n)$. We require that the reaction term f be locally Lipschitz in x and in u and converge to $-\infty$ at a super-linear rate as $u \to \infty$, for each $x \in R^n$. This latter requirement will be made more precise below.

Our first result is a sharp minimal growth condition on f, independent of L, in order that there exist a universal, a priori upper bound for all solutions to the Cauchy problem (1.1)—that is, in order that there exist a finite function $M(x,t)$ on $R^n \times (0,\infty)$ such that $u(x,t) \leq M(x,t)$, for all solutions to (1.1). After this result, we will always assume that $f(x,0) = 0$, so that $u \equiv 0$ is a solution to (1.1). We show that under a growth condition similar to the above one, an intimate relationship exists between two seemingly disparate phenomena—namely, uniqueness for the Cauchy problem (1.1) with initial data $g = 0$ and the nonexistence of unbounded, stationary solutions to the corresponding elliptic problem. We also give a generic condition for nonexistence of nontrivial stationary solutions.

For $R > 0$, define

$$F_R(u) = \sup_{|x| \leq R} f(x,u).$$

We will always assume that

$$\sup_{u>0} F_R(u) < \infty,\ \text{for all } R > 0. \tag{F-1}$$

Theorem 1 and Example 1 below show that the following assumption on F_R is a sharp condition for the existence of such a universal a priori upper bound, for all solutions to (1.1). Let $\log^{(n)} x$ denote the n-th iterate of $\log x$ so that $\log^{(1)} x = \log x, \log^{(2)} x = \log\log x$, etc.

$$\text{For each } R>0, \text{ there exist an } m \geq 0 \text{ and an } \epsilon > 0 \text{ such that}$$

$$\lim_{u\to\infty} \frac{F_R(u)}{u(\prod_{i=1}^m \log^{(i)} u)^2(\log^{(m+1)} u)^{2+\epsilon}} = -\infty, \tag{F-2}$$

where by convention, $\prod_{i=1}^0 \log^{(i)} = 1$.

Remark. F_R will satisfy (F-1) and (F-2) if, for instance, $f(x,u) = V(x)u - \gamma(x)u^p$, for $p > 1$, or if f is appropriately defined for small u and satisfies $f(x,u) = V(x)u - \gamma(x)u(\prod_{i=1}^m \log^{(i)} u)^2(\log^{(m+1)} u)^{2+\epsilon}$, for large u, where $V(x)$ is bounded on compacts and γ is positive and bounded away from 0 on compacts.

Theorem 1. *Assume that (F-1) and (F-2) hold. Then there exists a continuous function $M(x,t)$ on $R^n \times (0,\infty)$ such that every solution u to the Cauchy problem (1.1) satisfies $u(x,t) \leq M(x,t)$, for all $x \in R^n$ and all $t \geq 0$.*

The following example shows that condition (F-2) is sharp.

Example 1. Let $L = \frac{d^2}{dx^2}$ and $f(x,u) = -u\left((\log u)^2 + \log u\right)$, for $u \geq 1$. Then for each $l \in R$, $u_l(x) = \exp(\exp(x+l))$ solves (1.1) (as a stationary solution). Since $\lim_{l\to\infty} u_l(x) = \infty$, there is no universal a priori upper bound for all non-negative solutions of (1.1) for this choice of f. Alternatively, if we let $f(x,u) = -u\left((\log u)^2(\log\log u)^2 + \log u \log\log u\right)$, for $u \geq e$, then $u_l(x) = \exp(\exp(\exp(x+l)))$ solves (1.1). More generally, letting $u_l(x)$ denote the $(m+1)$-th iterate of the exponent function with argument $x+l$, then $Lu_l + f(u_l) = 0$, where the function f satisfies $f(u) < -u(\prod_{i=1}^m \log^{(i)} u)^2$, for large u

Remark. Consider the ordinary differential equation

$$v' = f(v),\ v(0) = c \geq 0, \tag{1.2}$$

where f is a Lipschitz function satisfying $f(0) = 0$ and $\lim_{u\to\infty} f(u) = -\infty$. The unique solution v_c to (1.2) satisfies $v_c \geq 0$ and is increasing as a function of its initial condition c. It is well-known and straight forward to show that $\lim_{c\to\infty} v_c(t) = \infty$, if $\int^\infty \frac{1}{-f(u)}du = \infty$, while $v_\infty(t) \equiv \lim_{c\to\infty} u_c(t) < \infty$, for $t > 0$, if $\int^\infty \frac{1}{-f(u)}du < \infty$. Thus, if the above integral is finite, v_∞ serves as a universal a priori upper bound for all solutions to (1.2), while if the above integral is infinite, there is no such finite function. In particular then, for the ordinary differential equation (1.2), a universal a priori upper bound on solutions exists when $f(u) = -u(\prod_{i=1}^m \log^{(i)} u)(\log^{(m+1)} u)^{1+\epsilon}$, but not when $f(u) = -u(\prod_{i=1}^m \log^{(i)} u)$. Comparing this with Theorem 1 and Example 1, one sees that *the introduction of spatial diffusion and drift slightly increases the minimal super-linearity threshold for the existence of a universal a priori upper bound.*

Define now

$$F(u) = \sup_{x\in R^n} f(x,u),$$

and consider the spatially uniform versions of conditions (F-1) and (F-2):

(F-1$'$)
$$\sup_{u>0} F(u) < \infty;$$

(F-2$'$)
$$\lim_{u\to\infty} \frac{F(u)}{u(\prod_{i=1}^m \log^{(i)} u)^2(\log^{(m+1)} u)^{2+\epsilon}} = -\infty, \text{for some } m \geq 0 \text{ and some } \epsilon > 0.$$

Consider also the following condition:

(F-3)
$$f(x,u) = 0 \text{ and } F(u) \text{ is locally Lipschitz.}$$

Remark. F will satisfy (F-1$'$), (F-2$'$) and (F-3) if, for instance, f is as in the remark following (F-2) with V bounded and γ positive and bounded away from 0.

The above conditions turn out to be critical for certain other important phenomena. Consider the associated elliptic equation corresponding to stationary solutions of (1.1):

$$
\begin{aligned}
&LW + f(x,W) = 0, \; x \in \; R^n; \\
&W \geq 0.
\end{aligned}
\tag{1.3}
$$

We will sometimes need one of the following two technical conditions on f:

(F-4a)
$G(u) \equiv \sup_{x\in R^n} \sup_{v\geq u}(f(x,v) - f(x,v-u))$ is locally Lipschitz, is negative for large u and satisfies $\int^{\infty} \frac{1}{-G(u)} du < \infty$.

(F-4b) $H(u) \equiv \sup_{x\in R^n} \sup_{v\geq 0}(f(x,u+v) - f(x,v))$ satisfies (F-1′) and (F-2′).

Remark. Note that if $f(x,\cdot)$ is concave for each $x \in R^n$ and F satisfies (F-1′), (F-2′) and (F-3), then both (F-4a) and (F-4b) hold. Indeed, by concavity, the supremum over v is attained in (F-4a) at $v = u$ and in (F-4b) at $v = 0$, giving $G(u) = H(u) = F(u) - F(0) = F(u)$. Furthermore, the integral condition in (F-4a) holds for any function satisfying (F-2′).

Theorem 2. *Assume that (F-1′), (F-2′), (F-3) and (F-4a) hold. If the trivial solution $u = 0$ is the only solution to the Cauchy problem (1.1) with initial data $g = 0$, then all solutions W to the stationary equation (1.3) are bounded. More specifically,*

$$W(x) \leq c_0, \; \textit{for all } x \in R^n,$$

where c_0 is the largest root of the equation $G(u) = 0$, and G is as in (F-4a). In particular, if $G(u) < 0$, for $u > 0$, then there are no nontrivial solutions to the stationary equation (1.3). (If $f(x,\cdot)$ is concave for each x, then (F-4a) is superfluous and $G = F$.)

Remark. Note that if under the condition in Theorem 2, one can exhibit an unbounded solution to the stationary equation (1.3), then Theorem 2 guarantees the existence of a nontrivial solution to the Cauchy problem (1.1) with 0 initial data. Examples 2 and 3 below are applications of this. Similarly, in the case that $G(u) < 0$, for $u > 0$, if one can exhibit a nontrivial solution to the stationary equation (1.3), then Theorem 2 guarantees the existence of a nontrivial solution to the Cauchy problem (1.1) with 0 initial data. An application of this is given on the top of page 9. These examples illustrate the utility of Theorem 2—it is much easier to construct appropriate solutions to (1.3) than to construct a nontrivial solution to (1.1) with initial data $g = 0$. By Theorem 2, with appropriate conditions on f, the existence of the latter is guaranteed by the existence of the former.

The next result gives conditions for uniqueness of the Cauchy problem (1.1) with initial data $g = 0$ and also for general initial data. Consider the following growth assumption on the coefficients of L:

$$\text{(L-1)} \qquad \begin{aligned} &\sum_{i,j=1}^{n} a_{ij}(x)\nu_i\nu_j \le C|\nu|^2(1+|x|^2) \\ &|b(x)| \le C(1+|x|), \end{aligned}$$

for some $C > 0$.

Theorem 3. *If (F-1′), (F-2′) (F-3) and (L-1) hold, then the trivial solution $u \equiv 0$ is the only solution to the Cauchy problem (1.1) with initial data $g = 0$. If in addition, (F-4b) holds, then there is a unique solution to the Cauchy problem (1.1) for each $g \in C(R^n)$.*

As immediate corollaries to Theorems 2 and 3, we obtain the following theorems.

Theorem 4. *If (F-1′), (F-2′), (F-3), (F-4a) and (L-1) hold, then all solutions to the stationary equation (1.3) are bounded.*

Theorem 5. *Assume that (F-1′), (F-2′), (F-3), (F-4a) and (L-1) hold. Assume in addition that the function G from condition (F-4a) satisfies $G(u) < 0$, for $u > 0$ (which will occur in particular if $f(x, \cdot)$ is concave for each x and $F(u) < 0$, for $u > 0$). Then there are no nontrivial solutions to the stationary equation (1.3).*

We elaborate now on Theorems 3 and 4 and then on Theorem 5. We begin by providing two examples which demonstrate that condition (L-1) is sharp for both Theorem 3 and Theorem 4.

Example 2. When $L = (1 + x^2)^{1+\epsilon} \frac{d^2}{dx^2}$ and $f(x, u) = -2u^{1+\epsilon}$, for some $\epsilon > 0$, then (1.3) possesses the unbounded solution $u(x) = 1 + x^2$. By Theorem 2, it then follows that there exists a nontrivial solution to (1.1) with initial data $g = 0$.

Example 3. When $L = \frac{d^2}{dx^2} + (1+x^2)^{\frac{1}{2}+\epsilon}\mathrm{sgn}(x)\frac{d}{dx}$ and $f(x, u) = -2-2(1+u)^{\frac{1}{2}+\epsilon}u^{\frac{1}{2}}$, for some $\epsilon > 0$, then (1.3) possesses the unbounded solution $u(x) = x^2$. By Theorem 2, it then follows that there exists a nontrivial solution to (1.1) with initial data $g = 0$.

Theorems 2-4 and Examples 2-3 suggest that under conditions (F-1′), (F-2′), (F-3) and the technical condition (F-4a), which is always satisfied if $f(x, \cdot)$ is concave, there may well be an equivalence between uniqueness for the Cauchy problem (1.1) with initial data $g = 0$ and nonexistence of unbounded solutions to the stationary equation (1.3). We leave this as an open problem.

Remark. We emphasize that in the context of this paper, uniqueness for the Cauchy problem (1.1) means uniqueness with regard to *all* classical, nonnegative solutions. If one works only with, say, mild solutions, then the situation can be quite different. For example, there is a unique mild solution for $u_t = \Delta u - \gamma(x)u^p$,

for $p > 1$ and bounded $\gamma \geq 0$ [8]; yet if γ decays sufficiently rapidly, uniqueness fails in the sense of all classical solutions (for details, see the next to the last paragraph in this section).

Turning to Theorem 5, it follows in particular that if $f(x, \cdot)$ is concave for each x, $F(u) < 0$, for $u > 0$, F satisfies conditions (F-2$'$) and (F-3), and the operator L satisfies condition (L-1), then there are no nontrivial solutions to (1.3). Generic results such as this regarding existence/nonexistence of nontrivial solutions to (1.3) seem to be rare in the literature. Indeed, the question of existence/nonexistence is delicate and can hinge greatly on the particular form of L and f. One generic result in the literature concerns the case that $f(\cdot, u)$ and the coefficients of L are periodic in x. Assume that for some $M_0 > 0$, $f(x, u) \leq 0$, for all $x \in R^n$ and all $u \geq M_0$. Let λ_0 denote the principal eigenvalue for the operator $L + \frac{\partial f}{\partial u}(x, 0)$ with periodic boundary conditions. If $\lambda_0 > 0$, then (1.3) possesses a nontrivial periodic solution, while if $\lambda_0 \leq 0$ and $\frac{f(x,u)}{u}$ is decreasing in u for each $x \in R^n$, then (1.3) does not possess a nontrivial *bounded* solution [1].

Consider now the well-studied case $L = \alpha(x)\Delta$ and $f(x, u) = -u^p$, with $p > 1$. If $n \geq 2$, then (1.3) possesses a nontrivial solution if $\lim_{|x|\to\infty} \frac{\alpha(x)}{(1+|x|)^{2+\epsilon}} > 0$, for some $\epsilon > 0$, and does not possess a nontrivial solution if $\lim_{|x|\to\infty} \frac{\alpha(x)}{(1+|x|)^2} < \infty$. For $n = 1$, the same result holds with the exponent 2 replaced by $1 + p$. For $n \geq 3$, this result goes back to [5] and [7], and it is shown in [5] that in the case of existence there are in fact an infinite number of bounded solutions. The n-dimensional analog of Example 2 above shows that there is also an unbounded solution. For $n = 1, 2$, the above result were proven in [2] and later apppeared with a different proof in [3] (which also re-derives the result for $n \geq 3$). Note that by Theorem 5, nonexistence of nontrivial solutions to (1.3) continues to hold for α in the above nonexistence range when the nonlinearity $-u^p$ is replaced by $f(x, u) = -u(\log(u + 1))^{2+\epsilon}$ or

$f(x,u) = -u(\log(u+e))^2(\log\log(u+e))^{2+\epsilon}$, etc., for some $\epsilon > 0$. Also, note that when α is in the above *existence* range, then by Theorem 2, there is a nontrivial solution to the Cauchy problem $u_t = \alpha\Delta u - u^p$ with initial condition $g = 0$.

An open problem was mentioned after Example 3. We now discuss some more open problems suggested by the above results and make some informal conjectures. Example 1 above shows that condition (F-2$'$) is sharp for Theorem 4. We don't believe that condition (F-2$'$) is sharp for Theorem 3. That is, we don't believe that the intimate connection between uniqueness for the Cauchy problem (1.1) with initial condition $g = 0$ and nonexistence of unbounded solutions to (1.3) continues to hold when condition (F-2$'$) is not in effect. Indeed, considering that uniqueness holds for positive solutions to the linear Cauchy problem $u_t = \Delta u - u$ and, by Theorem 3, also for the Cauchy problem $u_t = \Delta u + f(u)$, when f approaches $-\infty$ sufficiently fast so as to satisfy (F-2$'$), it seems likely that uniqueness also holds for $u_t = \Delta u + f(u)$ when f approaches $-\infty$ at a super-linear rate that does not satisfy (F-2$'$). We leave this as an open problem. If one replaces Δ with a general operator L, then the above heuristics become more uncertain. Indeed, for the linear equation $u_t = Lu - u$, uniqueness is known to hold when b satisfies the condition in (L-1) and when a satisfies a two-sided bound of the form $c(1+|x|^\gamma)|\nu|^2 \le \sum_{i,j=1}^n a_{ij}(x)\nu_i\nu_j \le C(1+|x|^\gamma)|\nu|^2$, for some $\gamma \in [0,2]$. It is not known whether uniqueness holds for the linear problem under condition (L-1). (For more about uniqueness in the linear case, see [4] and references therein.)

We also note that condition (F-2) in place of (F-2$'$) is not sufficient to insure uniqueness for the Cauchy problem (1.1). An example is given in the paragraph after next.

We believe that Theorem 2, and consequently Theorem 4, hold without the technical condition (F-4a). Similarly, Theorem 3 for nonzero initial data probably

holds without the technical condition (F-4b). These are also open problems.

We wish to emphasize an important point with regard to the connection between non-existence of unbounded solutions to (1.3) and uniqueness for the Cauchy problem (1.1) with initial condition $g = 0$. If one takes a scalar function $\alpha(x) > 0$ and replaces L and f by αL and αf respectively, then of course (1.3) remains unchanged. However, making the same change in the parabolic equation (1.1) can affect the question of uniqueness. Indeed, in [3], it was shown that if $L = \Delta$ and $f(x,u) = \frac{1}{\alpha}u^p$ with $p > 1$, then there exists a nontrivial solution to (1.1) with initial data $g = 0$ if $\alpha(x) \geq C\exp(|x|^{2+\epsilon})$, for some $\epsilon > 0$ and $C > 0$, and there doesn't exist such a solution if $\alpha(x) \leq C\exp(|x|^2)$, for some $C > 0$. (This is the example alluded to in the remark after Example 3 and in the paragraph before last.) On the other hand, for $L = \alpha\Delta$ and $f(x,u) = u^p$, it follows from Theorem 3 and from the final sentence in the paragraph on the top of page 9 that the existence of a nontrivial solution to (1.1) with initial data $g = 0$ depends on whether $\alpha(x) \geq C(1+|x|)^{l+\epsilon}$ or $\alpha(x) \leq C(1+|x|)^l$, where $l = 2$, if $n \geq 2$, and $l = 1+p$, if $n = 1$. What allows for the connection between uniqueness for the Cauchy problem (1.1) with initial condition $g = 0$ and nonexistence of solutions to the stationary equation (1.3) is the assumption (F-2$'$) on $F(u) = \sup_{x \in R^n} f(x,u)$. In the above example, this assumption requires one to consider $L = \alpha\Delta$ and $f(x,u) = u^p$, rather than $L = \Delta$ and $f(x,u) = \frac{1}{\alpha}u^p$.

Theorems 1 and 3 are proved by constructing appropriate super solutions, the proof of Theorem 3 being the much more delicate one. The proofs are given in sections two and three respectively. The proof of Theorem 2 uses a gamut of techniques and is given in section four.

2. Proof of Theorem 1. We begin with a standard maximum principle.

Proposition 1. *Let $D \subset R^n$ be a bounded domain and let* $0 \leq u_1, u_2 \in C^{2,1}(D \times$

$(0,\infty)) \cap C(\bar{D} \times [0,\infty))$ *satisfy*

$$Lu_1 + f(x,u_1) - \frac{\partial u_1}{\partial t} \le Lu_2 + f(x,u_2) - \frac{\partial u_2}{\partial t}, \text{ for } (x,t) \in D \times (0,\infty),$$

$$u_1(x,t) \ge u_2(x,t), \text{ for } (x,t) \in \partial D \times (0,\infty)$$

and

$$u_1(x,0) \ge u_2(x,0), \quad \text{for } x \in D.$$

Then $u_1 \ge u_2$ *in* $D \times (0,\infty)$.

Proof. Let $W = u_1 - u_2$ and define $V(x,t) = \frac{f(x,u_1(x,t)) - f(x,u_2(x,t))}{W(x,t)}$, if $W(x,t) \ne 0$, and $V(x,t) = 0$ otherwise. Since f is locally Lipschitz in u, V is bounded in $D \times [0,T]$, for any $T > 0$. We have $LW + VW - \frac{\partial W}{\partial t} \le 0$ in $D \times (0,\infty)$, $W(x,0) \ge 0$ in D, and $W(x,t) \ge 0$ on $\partial D \times (0,\infty)$. Thus, by the standard linear parabolic maximum principle, $u_1 \ge u_2$. $\square$

We record the following result, mentioned in the remark after Example 1.

Lemma 1. *Let* $G(u)$ *be Lipschitz and satisfy* $\lim_{u\to\infty} G(u) = -\infty$. *For* $c \ge 0$, *let* $v_c(t)$ *denote the solution to*

$$v' = G(v), \ t > 0; \qquad (2.1)$$
$$v(0) = c.$$

i. If $\int^\infty \frac{1}{-G(u)} du < \infty$, *then* $v_\infty(t) \equiv \lim_{c\to\infty} v_c(t) < \infty$, *for all* $t > 0$, *and* v_∞ *solves (2.1) with* $c = \infty$.

ii. If $\int^\infty \frac{1}{-G(u)} du = \infty$, *then* $\lim_{c\to\infty} v_c(t) = \infty$, *for all* $t \ge 0$.

Proof. We omit the straight forward proof of this standard result. $\square$

We now give the proof of Theorem 1. It suffices to show that for some $T_0 > 0$ and each $R > 0$, there exists a continuous function $M_R(x,t)$ on $\{|x| < R\} \times (0,T_0]$ such that every solution u to (1.1) satisfies $u(x,t) \le M_R(x,t) < \infty$, for $|x| < R$ and

$t \in (0, T_0]$. The reason it is enough to consider only $t \in (0, T_0]$ is that if $u(x,t)$ is a solution to (1.1), then $u(x, T_0 + t)$ is a solution to (1.1) with the initial condition $g(\cdot)$ replaced by $u(\cdot, T_0)$.

We will assume that F_R satisfies (F-2) with $m = 0$. At the end of the proof, we describe the simple change needed in the case that $m \geq 1$. In particular then, there exists an $\epsilon > 0$ and a $u_0 > 1$ such that

$$F_R(u) \leq -u(\log u)^{2+\epsilon} \equiv Q(u), \text{ for } u \geq u_0. \tag{2.2}$$

Since $\int^{\infty} \frac{1}{-Q(u)} du < \infty$, it follows from Lemma 1 that there exists a $T_0 > 0$ and a function $v_\infty(t)$ satisfying

$$\begin{aligned} &v'_\infty = Q(v_\infty), \ t \in (0, T_0]; \\ &v_\infty(0) = \infty; \\ &v_\infty(t) > 1, \ t \in (0, T_0]. \end{aligned} \tag{2.3}$$

Define

$$\phi_R(x) = \exp((R^2 - |x|^2)^{-l}),$$

with l satisfying $l\epsilon > 2$. Finally, choose K so that $\exp(K) > u_0$ and define

$$M_R(x,t) = \exp((K(t+1))\phi_R(x) + v_\infty(t).$$

Since $M_R(x,t) > u_0$, it follows from (2.2) that $f(x, M_R) \leq F_R(M_R) \leq Q(M_R)$. Since $Q(u)$ is concave for $u \geq 1$ and $Q(1) = 0$, it follows from the mean value theorem that $Q(b+a) - Q(b) < Q(a)$ for $1 \leq a \leq b$. Thus, since $\exp((K(t+1))\phi_R(x), v_\infty(t) > 1$, we have $Q(M_R(x,t)) < Q(\exp(K(t+1))\phi_R(x)) + Q(v_\infty(t))$. Using these facts along with (2.3), we obtain

(2.4)

$$\begin{aligned} &LM_R + f(x, M_R) - (M_R)_t \leq \exp(K(t+1))L\phi_R + Q(\exp(K(t+1))\phi_R) + Q(v_\infty) \\ &- K\exp(K(t+1))\phi_R - v'_\infty = \\ &\exp(K(t+1))L\phi_R + Q(\exp(K(t+1))\phi_R) - K\exp(K(t+1))\phi_R < \\ &\exp(K(t+1))\left(L\phi_R - (R^2 - |x|^2)^{-(2+\epsilon)l}\phi_R - K\phi_R\right), \text{ for } |x| < R \text{ and } t \in (0, T_0]. \end{aligned}$$

We have

$$\frac{L\phi_R(x)}{\phi_R(x)} = \big(4l^2(R^2-|x|^2)^{-2l-2} + 4l(l+1)(R^2-|x|^2)^{-l-2}\big)\sum_{i,j=1}^n a_{i,j}(x)x_ix_j \tag{2.5}$$
$$+ 2nl(R^2-|x|^2)^{-l-1} + 2l(R^2-|x|^2)^{-l-1}\sum_{i=1}^n x_ib_i(x).$$

The right hand side of (2.5) is bounded for $|x|$ in any ball of radius less than R. Furthermore, on the right hand side of (2.5), the dominating term as $|x| \to R$ is $4l^2(R^2-|x|^2)^{-2l-2}$. Thus, since $l\epsilon > 2$, it follows that the right hand side of (2.4) is negative if K is chosen sufficiently large. Using this with Proposition 1 and the fact that $M_R(x,0) = \infty$ and $M_R(x,t) = \infty$, for $|x| = R$, we conclude that any solution u to (1.1) satisfies $u(x,t) \le M_R(x,t)$, for $|x| < R$ and $t \in (0,T_0]$. This completes the proof of the theorem under the assumption that $m = 0$ in (F-2).

When $m > 0$ one simply replaces the test function $\phi_R(x)$ as above by $\phi_R(x) = \exp^{(m+1)}((R^2-|x|^2)^{-l})$, where $\exp^{(j)}$ denotes the j-th iterate of the exponential function. Everything goes through in a similar fashion.

□

3. Proof of Theorem 3. By assumption, $F(u) = \sup_{x\in R^n} f(x,u)$ satisfies (F-2′). As we did in the proof of Theorem 1, we will assume that $m = 0$ in (F-2′). At the appropriate point in the proof, we describe the simple change needed in the case that $m \ge 1$.

We first consider the case with initial condition $g = 0$. By conditions (F-2′) and (F-3), it follows that there exist $C_0, \epsilon > 0$ and $M_0 > 1$ such that

$$f(x,u) \le F(u) \le C_0u, \text{ for } u \le M_0; \tag{3.1}$$
$$f(x,u) \le F(u) \le -u(\log u)^{2+\epsilon}, \text{ for } u \ge M_0.$$

Fix $R > 1$ and $T \in (0,\infty)$. Define

$$\phi_R(x) = \exp\Big(\big(\frac{1+|x|^2}{R^2-|x|^2}\big)^l\Big),$$

with l satisfying $l\epsilon > 2$, and define

$$\psi_R(x,t) = (\phi_R(x) - 1)\exp(K(t+1)),$$

with $K > 0$. A direct calculation reveals that

$$L\psi_R = \exp(K(t+1))\phi_R(x)\left[W_1 + W_2 + W_3 + W_4 + W_5\right], \tag{3.2}$$

where

$$\begin{aligned}
W_1 &= 4l^2(1+|x|^2)^{2l-2}(R^2-|x|^2)^{-2l-2}(R^2+1)^2\sum_{i,j=1}^n a_{i,j}(x)x_ix_j;\\
W_2 &= 4l(l-1)(1+|x|^2)^{l-2}(R^2-|x|^2)^{-l-2}(R^2+1)^2\sum_{i,j=1}^n a_{i,j}(x)x_ix_j;\\
W_3 &= 4l(1+|x|^2)^{l-1}(R^2-|x|^2)^{-l-2}(R^2+1)\sum_{i,j=1}^n a_{i,j}(x)x_ix_j;\\
W_4 &= 2nl(1+|x|^2)^{l-1}(R^2-|x|^2)^{-l-1}(R^2+1);\\
W_5 &= 2l(1+|x|^2)^{l-1}(R^2-|x|^2)^{-l-1}(R^2+1)\sum_{i=1}^n x_ib_i.
\end{aligned}$$

We also have

$$\frac{\partial\psi_R}{\partial t} = K\psi_R(x). \tag{3.3}$$

We claim that for K sufficiently large and independent of R (but not independent of T in (3.4-b))

$$\begin{aligned}
&\exp(K(t+1))\phi_R(x)W_i - \frac{1}{5}(K-C_0)\psi_R(x,t) \le 0,\\
&\text{if } \psi_R(x,t) \le M_0, \text{ for } |x| < R \text{ and } t \in (0,T];
\end{aligned} \tag{3.4-a}$$

$$\begin{aligned}
&\exp(K(t+1))\phi_R(x)W_i - \frac{1}{5}K\psi_R(x,t) - \frac{1}{5}\psi_R(x,t)(\log\psi_R(x,t))^{2+\epsilon} \le 0,\\
&\text{if } \psi_R(x,t) \ge M_0, \text{ for } |x| < R \text{ and } t \in (0,T],
\end{aligned} \tag{3.4-b}$$

for $i = 1, 2, 3, 4, 5$

From (3.1)-(3.4), it follows that for sufficiently large K, independent of R,

$$L\psi_R - \frac{\partial \psi_R}{\partial t} + f(x, \psi_R) \leq 0, \text{ for } |x| < R \text{ and } t \in (0, T]. \tag{3.5}$$

Since $\psi_R(x, 0) \geq 0$ and $\lim_{|x| \to R} \psi_R(x, t) = \infty$, it follows from (3.5) and the maximum principle in Proposition 1 that any solution u to (1.1) with initial condition $g = 0$ must satisfy the bound

$$u(x,t) \leq (\exp((\frac{1 + |x|^2}{R^2 - |x|^2})^l) - 1) \exp(K(t + 1)), \text{ for } |x| < R \text{ and } t \in (0, T]. \tag{3.6}$$

Since K doesn't depend on R, letting $R \to \infty$ in (3.6) gives $u(x, t) \equiv 0$ for $x \in R^n$ and $t \in (0, T]$. Now letting $T \to \infty$ gives $u(x, t) \equiv 0$ in $R^n \times (0, \infty)$, completing the proof.

When $m > 0$ one replaces the test function $\phi_R(x)$ as above by $\phi_R(x) = \exp^{(m)}((\frac{1+|x|^2}{R^2-|x|^2})^l)$, where $\exp^{(j)}$ denotes the j-th iterate of the exponential function. The resulting calculations are similar to the present case.

It thus remains to prove (3.4) for K independent of R. We will prove (3.4) for W_1. The proofs for $W_i, i \geq 2$, are similar. Consider first (3.4-a). We will always assume that $K \geq C_0$. Recall the definitions of ϕ_R and ψ_R. If $\psi_R(x, t) \leq M_0$, then a fortiori $\phi_R(x) \leq M_0 + 1$ and $(\frac{1+|x|^2}{R^2-|x|^2})^l \leq \log(M_0 + 1) \equiv L_0^l$. Also, we have $\psi_R(x, t) \geq \phi_R(x) - 1 \geq (\frac{1+|x|^2}{R^2-|x|^2})^l$. In light of these observations, it follows that (3.4-a) will hold if

$$(M_0 + 1)W_1 - \frac{1}{5}(K - C_0)(\frac{1 + |x|^2}{R^2 - |x|^2})^l \leq 0, \text{ whenever } \frac{1 + |x|^2}{R^2 - |x|^2} \leq L_0.$$

Or equivalently, if

$$K \geq C_0 + 5(\frac{R^2 - |x|^2}{1 + |x|^2})^l (M_0 + 1)W_1, \text{ whenever } \frac{1 + |x|^2}{R^2 - |x|^2} \leq L_0. \tag{3.7}$$

Thus, we must show that the right hand side of (3.7) is bounded in R and x under the constraint $\frac{1+|x|^2}{R^2-|x|^2} \leq L_0$. Substituting for W_1 in the right hand side of (3.7) and using the assumption that $\sum_{i,j=1}^n a_{i,j}(x)x_ix_j \leq C(1+|x|^2)|x|^2$, one finds that it is enough to show that $(\frac{1+|x|^2}{R^2-|x|^2})^{l-1}\frac{(R^2+1)^2|x|^2}{(R^2-|x|^2)^3}$ is bounded in R and x under the above constraint. Since $(\frac{1+|x|^2}{R^2-|x|^2})^{l-1}$ is trivially bounded under the constraint, it remains only to consider $\frac{(R^2+1)^2|x|^2}{(R^2-|x|^2)^3}$. The constraint above is equivalent to the constraint $|x|^2 \leq \frac{L_0R^2-1}{L_0+1}$. From this it is clear that under the constraint, $\frac{(R^2+1)^2|x|^2}{(R^2-|x|^2)^3}$ is bounded in R and x.

We now turn to (3.4-b). The constraint $\psi_R(x,t) \geq M_0$ along with the condition $t \leq T$ guarantee the existence of a $c_0 \in (0,1)$ such that $\phi_R(x) - 1 \geq c_0\phi_R(x)$. Note that c_0 depends on T, but not on R. Thus, under the constraint, we have $\psi(x,t) = (\phi_R(x)-1)\exp(K(t+1)) \geq c_0\phi_R(x)\exp(K(t+1))$. Therefore (3.4-b) will hold if we show that K can be picked independent of R and such that $W_1 - \frac{1}{5}c_0K - \frac{1}{5}c_0\left(\log\phi_R + \log c_0 + K(t+1)\right)^{2+\epsilon} \leq 0$ holds under the constraint. We will always assume that $K \geq -\log c_0$. Thus it suffices to show that $W_1 - \frac{1}{5}c_0(\log\phi_R)^{2+\epsilon}$ is bounded from above under the constraint, independent of R. Substituting for ϕ_R and W_1 and using the assumption $\sum_{i,j=1}^n a_{i,j}(x)x_ix_j \leq C(1+|x|^2)|x|^2$, it is sufficient to show that $4l^2C(1+|x|^2)^{2l-1}(R^2-|x|^2)^{-2l-2}(R^2+1)^2|x|^2 - \frac{1}{5}c_0\left(\frac{1+|x|^2}{R^2-|x|^2}\right)^{(2+\epsilon)l}$ is bounded from above under the constraint, or equivalently, that

$$\left(\frac{1+|x|^2}{R^2-|x|^2}\right)^{(2+\epsilon)l}\left(4l^2C\frac{(R^2+1)^2|x|^2}{(1+|x|^2)^3}\left(\frac{R^2-|x|^2}{1+|x|^2}\right)^{l\epsilon-2} - \frac{c_0}{5}\right) \tag{3.8}$$

is bounded from above under the constraint.

We may assume that $4l^2C\frac{(R^2+1)^2|x|^2}{(1+|x|^2)^3}\left(\frac{R^2-|x|^2}{1+|x|^2}\right)^{l\epsilon-2} \geq \frac{c_0}{5}$, since otherwise it is clear that (3.8) holds. From this inequality and the assumption that $l\epsilon > 2$, it follows that

$$\frac{1+|x|^2}{R^2-|x|^2} \leq \left(\frac{20l^2C}{c_0}\frac{(R^2+1)^2|x|^2}{(1+|x|^2)^3}\right)^{\frac{1}{l\epsilon-2}}. \tag{3.9}$$

Furthermore, the constraint $\psi_R \geq M_0$ guarantees that

$$\frac{1+|x|^2}{R^2-|x|^2} \geq (\log(1+M_0\exp(-K(T+1))))^{\frac{1}{l}} \equiv \gamma_0 > 0, \tag{3.10}$$

which can be written in the form

$$|x|^2 \geq \frac{\gamma_0 R^2 - 1}{\gamma_0 + 1}. \tag{3.11}$$

If $|x|$ satisfies (3.11), then the right hand side of (3.9) is bounded. Therefore, in (3.8), the terms $(\frac{1+|x|^2}{R^2-|x|^2})^{(2+\epsilon)l}$ and $\frac{(R^2+1)^2|x|^2}{(1+|x|^2)^3}$ are bounded. And by (3.10), the term $(\frac{R^2-|x|^2}{1+|x|^2})^{l\epsilon-2}$ is also bounded. This completes the proof of (3.8).

We now turn to the case that the initial condition g is not equal to 0. We assume now in addition that condition (F-4b) is in effect. Fix $R > 1$ and $T \in (0,\infty)$. Let $\psi_R(x,t)$ be as in part (i), but corresponding to the function H appearing in condition (F-4b), rather than corresponding to the function F as in part (i).

In [3], for the case $f(x,u) = V(x)u - \gamma(x)u^p$, we showed that there exists a minimal solution u_g to (1.1); that is, a solution u_g with the property that $u_g(x,t) \leq u(x,t)$, for any solution u to (1.1) with initial data g. In fact, the proofs there go through for general locally Lipschitz continuous f as long as a universal a priori upper bound exists. Thus, in light of Theorem 1, there exists a minimal solution u_g. (In fact, u_g is obtained by taking the solution of (4.1) below and letting $m \to \infty$.)

Now define $\hat{\psi}_R(x,t) = \psi_R(x,t) + u_g$. Then

$$\begin{aligned} L\hat{\psi}_R + f(x,\hat{\psi}_R) - \frac{\partial \hat{\psi}_R}{\partial t} &= (L\psi_R + H(\psi_R) - \frac{\partial \psi_R}{\partial t}) \\ + (Lu_g + f(x,u_g) - \frac{\partial u_g}{\partial t}) &+ (f(x,\psi_R+u_g) - f(x,u_g) - H(\psi_R)). \end{aligned} \tag{3.12}$$

The first of the three terms on the right hand side of (3.12) is non-positive by the construction in part (i), the second term is non-positive because u_g is a solution to (1.1), and the third term is non-positive by the definition of H in (F-4b). The

argument used above for part (i) in the paragraph in which (3.5) appears then shows that any solution u to (1.1) must satisfy $u(x,t) \le u_g(x,t) + \psi_R(x,t)$, for $|x| < R$ and $t \in (0,T]$. Letting $R \to \infty$ and then $T \to \infty$ as before shows that $u = u_g$. $\square$

4. Proof of Theorem 2. We need to utilize certain constructions that were carried out in [3, section 2] for the case that $f(x,u) = V(x) - \gamma(x)u^p$. These constructions are based on results in [6], and hold with the same proofs for general locally Lipschitz continuous f as long as a universal a priori upper bound exists. Thus, in light of Theorem 1, they hold for f satisfying (F-1) and (F-2).

Let $B_m \subset R^n$ denote the open ball of radius m centered at the origin. There exists a solution $u \in C^{2,1}(B_m \times (0,\infty)) \cap C(B_m \times [0,\infty)) \cap C(\bar{B}_m \times (0,\infty))$ to the equation

$$
\begin{aligned}
&u_t = Lu + f(x,u),\ (x,t) \in B_m \times (0,\infty);\\
&u(x,0) = g(x),\ x \in B_m;\\
&u(x,t) = 0,\ (x,t) \in \partial B_m \times (0,\infty),
\end{aligned}
\tag{4.1}
$$

for any $0 \le g \in C(\bar{B}_m)$. (See the beginning of the proof of Theorem 1 in [3], where the above construction is first made in the case that g is compactly supported in B_m, and then extended to the case that $g \in C(\bar{B}_m)$.)

Now let W be an arbitrary solution to (1.3). For $m > 0$ and a positive integer k, let $\psi_{m,k} \in C^\infty(R^n)$ satisfy

$$
\begin{gathered}
\psi_{m,k}(x) = 0,\ |x| \le m \text{ and } |x| > 2m+1\\
\psi_{m,k}(x) = k,\ m+1 \le |x| \le 2m\\
0 \le \psi_{m,k} \le k.
\end{gathered}
$$

There exists a nonnegative solution $U_{m,k} \in C^{2,1}(B_{2m} \times (0,\infty)) \cap C(\bar{B}_{2m} \times (0,\infty))$

to the equation

$$
\begin{aligned}
&u_t = Lu + f(x,u) + \psi_{m,k}, \ (x,t) \in B_{2m} \times (0,\infty);\\
(4.2) \qquad &u(x,0) = g_m, \ x \in B_{2m};\\
&u(x,t) = 0, \ (x,t) \in \partial B_{2m} \times (0,\infty),
\end{aligned}
$$

where $g_m \geq 0$ is continuous and satisfies

$$g_m(x) = \begin{cases} 0, \text{ for } \ x \in B_m \\ m^2 W, \text{ for } x \in B_{2m} - B_{m+1} \end{cases}.$$

(This construction is similar to the one in [3,equation (2.5)].) Also,

(4.3)

$$U(x,t) \equiv \lim_{m\to\infty} \lim_{k\to\infty} U_{m,k}(x,t) \text{ is a solution to (1.1) with initial condition } g = 0.$$

(See the two paragraphs after equation (2.6) in [3], ignoring equation (2.6) and the concept of a maximal solution that appears there.)

Consider (4.1) with m replaced by $2m$, with the nonlinearity f replaced by G as in condition (F-4a), and with $g = W$. Denote the solution to this equation by u_m. We will show below that

$$(4.4) \ W - u_m \leq U_{m,k}, \text{ in } B_{\frac{3m}{2}} \times [0,\infty), \text{for } k \text{ sufficiently large, depending on } \ m.$$

Let v_∞ denote the solution to $v' = G(v)$ with $v_\infty(0) = \infty$, as in Lemma 1-i (note that by condition (F-4a), G satisfies the requirement in Lemma 1-i). Since $f(x,u) \leq G(u)$, we have $f(x,v_\infty) - v'_\infty \leq G(v_\infty) - v'_\infty = 0$. Also, since G is Lipschitz, it follows from the uniqueness theorem for ordinary differential equations that $v_\infty(t) > 0$, for all $t \geq 0$. Using these facts along with the fact that $u_m = 0$ on ∂B_{2m} and the fact that $v_\infty(0) = \infty$, it follows from the maximum principle in Proposition 1 that

$$(4.5) \qquad u_m(x,t) \leq v_\infty(t) \text{ in } B_{2m} \times (0,\infty).$$

Letting $k \to \infty$ and then letting $m \to \infty$, it follows from (4.3) that the right hand side of (4.4) converges to a solution U of (1.1) with initial data $g = 0$. By the uniqueness assumption, $U = 0$. Using this with (4.5) then gives

$$W(x) \leq v_\infty(t) \text{ in } R^n \times (0, \infty). \tag{4.6}$$

We now show that

$$\lim_{t \to \infty} v_\infty(t) = c_0, \text{ where } c_0 \text{ is the largest root of } G(u) = 0. \tag{4.7}$$

To see this, let v_c be as in Lemma 1. Integrating, changing variables and letting $c \to \infty$, we obtain

$$\int_{v_\infty(t)}^{\infty} \frac{1}{-G(u)} du = t. \tag{4.8}$$

Letting $t \to \infty$ in (4.8) and using the fact that G is locally Lipschitz proves (4.7). The theorem now follows from (4.6) and (4.7).

It remains to prove (4.4). Let $V = W - u_m$. We have

$$\begin{aligned} &LV + f(x, V) - V_t = (LW + f(x, W)) - (Lu_m + G(u_m) - (u_m)_t) + \\ &f(x, W - u_m) - f(x, W) + G(u_m) = \\ &f(x, W - u_m) - f(x, W) + G(u_m) \geq 0 \text{ in } B_{2m} \times (0, \infty), \end{aligned} \tag{4.9}$$

where the second equality follows from the definitions of W and u_m, and the inequality follows from the definition of G. On the other hand, we have

$$LU_{m,k} + f(x, U_{m,k}) - (U_{m,k})_t = -\psi_{m,k} \leq 0 \text{ in } B_{2m} \times (0, \infty). \tag{4.10}$$

We now show that for sufficiently large k, depending on m,

$$V(x, t) \leq U_{m,k}(x, t), \text{ on } \partial B_{\frac{3m}{2}} \times [0, \infty). \tag{4.11}$$

Define $Q(x) = \big(l^2 - (m+1+l-|x|)^2\big)\, W(x)$, where $l = \frac{1}{2}(m-1)$. Note that $Q > 0$ in the annulus $A_{m+1,2m} \equiv \{m+1 < |x| < 2m\}$ and vanishes on $\partial A_{m+1,2m}$. Clearly $LQ + f(x,Q)$ is bounded in $A_{m+1,2m} \times [0,\infty)$. Thus for k sufficiently large, we have $LQ + f(x,Q) \geq -\psi_{m,k}$ in $A_{m+1,2m} \times [0,T]$. Since $Q(x) \leq l^2 W(x) < m^2 W(x) = g_m(x) = U_{m,k}(x,0)$ on $A_{m+1,2m}$, and since Q vanishes on $\partial A_{m+1,2m}$, it follows by the maximum principle in Proposition 1 that $U_{m,k} \geq Q$ in $A_{m+1,2m} \times [0,\infty)$, for k sufficiently large. Substituting $|x| = \frac{3m}{2}$ in Q, we conclude that for $m \geq 4$ and sufficiently large k, $U_{m,k}(x,t) \geq Q(x) = (l^2 - \frac{1}{4})W(x) > W(x)$ on $\partial B_{\frac{3m}{2}} \times [0,\infty)$. This proves (4.11) since $V \leq W$. In light of (4.9)-(4.11) and the fact that $V(x,0) = 0$, (4.4) now follows from the maximum principle in Proposition 1. $\square$

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